Teoria Espectral

1. Spectral Theorem

Here we are interested in extending the spectral theorem from some bounded linear operators to self-adjoint unbounded linear operators. We are going to give most of the main details to establish the *Spectral Measure Version of the Spectral Theorem*. We will also give some details of the *Multiplication Operator Form of the Spectral Theorem*. We follow the notes by Bernard Helffer [1], the books by Reed and Simon [3, 4] and class notes.

We will start by recalling the spectral theorem for compact operators.

Theorem 1.1. *Let $\mathcal{H}$ be a separable Hilbert space and T a compact self-adjoint operator. Then $\mathcal{H}$ admits a Hilbertian basis consisting of the eigenfunctions of T.*

More precisely, we can obtain a decomposition of $\mathcal{H}$ in the form

$$\mathcal{H} = \bigoplus_{k \in \mathbb{N}} V_k$$

such that

$$T u_k = \lambda_k \, u_k, \quad \text{if } u_k \in V_k$$

Thus $\mathcal{H}$ has been decomposed into a direct sum of orthogonal subspaces V_k in which the self-adjoint operator T is reduced to multiplication by λ_k.

We recall that an operator $P \in \mathcal{B}(\mathcal{H})$ is called an **orthogonal projection** if $P = P^*$ and $P^2 = P$.

If P_k denotes the orthogonal projection operator onto V_k, we can write

$$I = \sum_k P_k \qquad \text{(the limit is in the strong convergence sense)}$$

and

$$Tu = \sum_k \lambda_k P_k u, \quad \forall u \in D(T).$$

This decomposition is the inspiration to extend the spectral theorem for self-adjoint unbounded operators as we will see below.

1.1. **Spectral family and resolution of the identity.**

Definition 1.2. *A family of orthogonal projectors $E(\lambda)$ (or E_λ), $-\infty < \lambda < \infty$ in a Hilbert space $\mathcal{H}$ is called a* **resolution of the identity** *(or* **spectral family***) if it satisfies the following conditions:*

(i)

$$E(\lambda)E(\mu) = E(\min(\lambda, \mu)), \tag{1.1}$$

(ii)

$$E(-\infty) = 0, \qquad E(+\infty) = I \tag{1.2}$$

where $E(\pm\infty)$ is defined by

$$E(\pm\infty)x = \lim_{\lambda\to\pm\infty} E(\lambda)x \quad \text{for all } x \in \mathcal{H}, \tag{1.3}$$

(iii)

$$E(\lambda + 0) = E(\lambda) \tag{1.4}$$

where $E(\lambda + 0)$ is defined by

$$E(\lambda + 0)x = \lim_{\substack{\mu>\lambda \\ \mu\to\lambda}} E(\mu)x \quad \text{for all } x \in \mathcal{H}. \tag{1.5}$$

Remark 1.3. *Observe that* (1.1) *gives the existence of the limit. The limit in* (1.3) *is taken in $\mathcal{H}$. We also notice that $\lambda \mapsto \langle E(\lambda)x, x\rangle = \|E(\lambda)x\|^2$ is monotonically increasing.*

Consider the spectral family $E(\lambda)$. The following statements are equivalent.

1.

$$\|E(\lambda)\varphi\| \leq \|E(\mu)\varphi\| \qquad \lambda \leq \mu. \tag{1.6}$$

2.

$$E(\lambda) = E(\lambda)E(\mu) = E(\mu)E(\lambda) \qquad \lambda \leq \mu. \tag{1.7}$$

Proof. If $E_\mu\phi = 0$ from (1.6) we deduce that $E_\lambda\phi = 0$.

On the other hand,

$$E_\mu\underbrace{(E_\mu - I)\varphi}_{\psi} = 0, \qquad \forall\varphi \in \mathcal{H}.$$

Hence

$$E_\lambda\psi = 0 \iff E_\lambda(E_\mu - I)\varphi = 0.$$

Thus

$$E_\lambda E_\mu = E_\lambda.$$

In other words, if $\varphi \in \operatorname{Ker} E(\lambda)$, then

$$E_\lambda \varphi = E_\lambda E_\mu \varphi$$

Now suppose that $\varphi \in E(\lambda)\mathcal{H}$. First we notice that $\varphi = E(\lambda)\varphi$. Hence

$$\begin{aligned}
\|\varphi\| = \|E(\lambda)\varphi\| &\leq \|E(\mu)\varphi\| = \|E(\mu)E(\lambda)\varphi\| \\
&\leq \|E(\mu)E(\lambda)\|\|\varphi\| \\
&\leq \|E(\mu)\|\|E(\lambda)\|\|\varphi\| \\
&= \|\varphi\|
\end{aligned}$$

where we have used that the norm of a orthogonal projection is 1. Thus

$$\|\varphi\| = \|E(\mu)\varphi\|. \tag{1.8}$$

The identity (1.8) implies that

$$\|\varphi\|^2 = \|E(\mu)\varphi\|^2 + \|(I - E(\mu))\varphi\|^2 = \|E(\mu)\varphi\|^2.$$

This implies that

$$\|(I - E(\mu))\varphi\| = 0$$

and so $E(\mu)\varphi = \varphi$. Thus

$$E(\mu)E(\lambda)\varphi = E(\lambda)\varphi.$$

□

Remark 1.4. *The property* (1.6) *implies that*

$$s - \lim_{\substack{\lambda \to \mu \\ \lambda < \mu}} E(\lambda) = E(\mu - 0)$$

exists.

Indeed,

$$\begin{aligned}
\|E(\mu)\varphi - E(\lambda)\varphi\|^2 &= \|E(\mu)\varphi\|^2 + \|E(\lambda)\varphi\|^2 - 2\|E(\lambda)\varphi\|^2 \\
&= \|E(\mu)\varphi\|^2 - \|E(\lambda)\varphi\|^2.
\end{aligned}$$

From (1.6) *we deduce that*

$$\lim \|E(\lambda)\varphi\| = \sup_{(-\infty,\mu]} \|E(\lambda)\varphi\| = \|E(\mu)\varphi\|$$

It follows then that

$$\lim_{\substack{\lambda \to \mu \\ \lambda < \mu}} \|(E(\mu) - E(\lambda))\varphi\| = 0.$$

Example 1.5 (Spectral family associated to H_0)**.** *Let*

$$\mathrm{G}_\lambda(|\xi|^2) = \begin{cases} 0 & \text{if } \lambda < 0, \\ \chi_{\{|\xi|^2<\lambda\}} & \text{if } \lambda \geq 0. \end{cases}$$

$$\mathrm{G}_\lambda(H_0)f = \mathcal{F}^{-1}[G_\lambda(|\cdot|^2)\mathcal{F}f] = (\mathrm{G}_\lambda(|\xi|^2)\widehat{f}\,)^\vee = E_0(\lambda)f$$

- $\lim\limits_{\lambda\to 0} E_0(\lambda)f = 0, \quad \big(\lim\limits_{\lambda\to -\infty} E_0(\lambda)f = 0\big) \quad \forall f \in L^2(d\xi)$
- $\lim\limits_{\lambda\to\infty} E_0(\lambda)f = f \qquad \forall f \in L^2(d\xi)$
- $E_0(\lambda)$ *is an orthonormal projection for any* λ.
- $E_0(\lambda)^2 = E_0(\lambda)$
- $E_0(\lambda) = E_0^*(\lambda)$
- $$\begin{aligned} \|E_0(\lambda)f\|^2 &= (E_0(\lambda)f, E_0(\lambda)f) \\ &= (E_0^2(\lambda)f, f) \\ &= (E_0(\lambda)f, f). \end{aligned}$$

Exercise 1.6. *Prove that* $E(\lambda) = M_{\chi_{(-\infty,\lambda]}}$ *is a spectral family,*

$$(E(\lambda)\varphi|\psi) = \int_{-\infty}^{\lambda} \varphi(x)\,\overline{\psi(x)}\,dx$$

and

$$\frac{d}{d\lambda}(E(\lambda)\varphi|\psi) = \varphi(\lambda)\,\overline{\psi(\lambda)} \qquad \text{a.e. } \lambda.$$

That is, we have a measure $\mu_{\varphi,\psi}$ *such that*

$$d\mu_{\varphi,\psi} = \varphi(\lambda)\,\overline{\psi(\lambda)}\,d\lambda,$$

or

$$d(E(\lambda)\varphi|\psi) = \varphi(\lambda)\,\overline{\psi(\lambda)}\,d\lambda.$$

We observe that

$$\begin{aligned} \int_{\mathbb{R}} \lambda\, d(E(\lambda)\varphi|\psi) &= \int_{\mathbb{R}} \lambda\varphi(\lambda)\,\overline{\psi(\lambda)}\,d\lambda \\ &= \int_{\mathbb{R}} x\,\varphi(x)\,\overline{\psi(x)}\,dx < \infty \quad \textit{whenever } \; x\varphi(x) \in L^2(\mathbb{R}) \end{aligned}$$

Hence for

$$D(M) = \{\varphi \in L^2(\mathbb{R}) : x\varphi \in L^2(\mathbb{R})\}$$

$$M\varphi = x\varphi$$

we have

$$\big(M\varphi|\psi\big) = \int \lambda\, d(E(\lambda)\varphi\,|\psi)\, d\lambda.$$

Proposition 1.7. *Let $E(\lambda)$ be a resolution of identity; then for all $x, y \in \mathcal{H}$, the function*

$$\lambda \mapsto \langle E(\lambda)x, y\rangle \tag{1.9}$$

is a function of bounded variation whose total variation satisfies

$$V(x,y) \leq \|x\| \cdot \|y\|, \quad \forall x, y \in \mathcal{H}. \tag{1.10}$$

where

$$V(x,y) = \sup_{\lambda_1,\dots,\lambda_n} \sum_{j=2}^{n} \big|\langle E_{(\lambda_{j-1},\lambda_j]}x, y\rangle\big|. \tag{1.11}$$

Proof. Let $\lambda_1 < \lambda_2 < \cdots < \lambda_n$. From the assumption (1.1) we deduce that

$$E_{(\alpha,\beta]} = E_\beta - E_\alpha$$

is an orthogonal projection. The Cauchy-Schwarz inequality yields

$$\begin{aligned}
\sum_{j=2}^{n} |\langle E_{(\lambda_{j-1},\lambda_j]}x, y\rangle| &= \sum_{j=2}^{n} |\langle E_{(\lambda_{j-1},\lambda_j]}x, E_{(\lambda_{j-1},\lambda_j]}y\rangle| \\
&\leq \sum_{j=2}^{n} \|E_{(\lambda_{j-1},\lambda_j]}x\| \|E_{(\lambda_{j-1},\lambda_j]}y\| \\
&\leq \Big(\sum_{j=2}^{n} \|E_{(\lambda_{j-1},\lambda_j]}x\|^2\Big)^{1/2} \Big(\sum_{j=2}^{n} \|E_{(\lambda_{j-1},\lambda_j]}y\|^2\Big)^{1/2} \\
&= \big(\|E_{(\lambda_1,\lambda_n]}x\|^2\big)^{1/2} \big(\|E_{(\lambda_1,\lambda_n]}y\|^2\big)^{1/2}.
\end{aligned}$$

Then for $m > n$, we obtain

$$\|x\|^2 \geq \|E_{(\lambda_n,\lambda_m]}x\|^2 = \sum_{i=n}^{m-1} \|E_{(\lambda_i,\lambda_{i+1}]}x\|^2.$$

Thus for any finite sequence $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ we have

$$\sum_{j=2}^{m} |\langle E_{(\lambda_{j-1},\lambda_j]}x, y\rangle| \leq \|x\| \|y\|.$$

Using (1.11), the estimate (1.10) follows. $\square$

We have proved that for all x and y in $\mathcal{H}$, the function $\lambda \mapsto \langle E(\lambda)x, y\rangle$ is with bounded variation and we can then show the existence of $E(\lambda + 0)$ and $E(\lambda - 0)$. Indeed, the following lemma regards this.

Lemma 1.8. *If $E(\lambda)$ is a family of projectors satisfying* (1.1) *and* (1.2)*, then for all $\lambda \in \mathbb{R}$, the operators*

$$E_{\lambda+0} = \lim_{\mu\to\lambda,\, \mu>\lambda} E(\mu) \quad \textit{and} \quad E_{\lambda-0} = \lim_{\mu\to\lambda,\, \mu<\lambda} E(\mu) \tag{1.12}$$

are well defined when considering the limit for the strong convergence topology.

Proof. We prove the existence of the left limit. Using (1.1), we deduce that for any $\epsilon > 0$, there exists $\lambda_0 < \lambda$ such that, $\forall\lambda', \forall\lambda'' \in [\lambda_0, \lambda)$ with $\lambda' < \lambda''$

$$\|E_{(\lambda',\lambda'']}x\|^2 \leq \epsilon.$$

It is not difficult to prove that $E_{\lambda-\frac{1}{n}}x$ is a Cauchy sequence converging to a limit and that limit does not depend on the sequence going to λ.

A similar argument shows the existence of the limit from the right. $\square$

1.2. **Spectral Integrals.** It is then classical (Stieltjes integrals) that one can define for any continuous complex valued function $\lambda \mapsto f(\lambda)$ the integrals

$$\int_a^b f(\lambda)\, d\langle E(\lambda)x, y\rangle$$

as a limit of Riemann sums.

Indeed, let $\mathcal{H}$ be a Hilbert space. Consider a function $F : [a, b] \to \mathbb{C}$, $E(\lambda)$ a spectral family, and the partition

$$a \leq \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_n = b.$$

Let $\Delta_j = (\lambda_{j-1}, \lambda_j]$, then

$$\begin{aligned} E(\Delta_j) &:= E(\lambda_j - 0) - E(\lambda_{j-1} + 0) \\ &= E(\lambda_j) - E(\lambda_{j-1}). \end{aligned}$$

We define the Riemann sums

$$\mathsf{S}(F, \pi, \{\xi_j\})\varphi = \sum_{j=1}^{n} F(\xi_j)\,(E_{(\lambda_{j-1},\lambda_j]})\varphi$$

where $\xi_j \in \Delta_j$ and π is a partition of $[a, b]$.

Definition 1.9. *We say that F is integrable with respect to $E(\lambda)\varphi$ if and only if*

$$\lim_{\|\pi\|\to 0} \mathsf{S}(F,\pi,\{\xi_j\})\varphi$$

exists independently of the choice of the points of the partition ξ_j. In this case

$$\lim_{\|\pi\|\to 0} \mathsf{S}(F,\pi,\{\xi_j\})\varphi := \int_a^b F(\lambda)\, d\, E(\lambda)\varphi.$$

If the limit exist for all φ we will have a bounded linear operator given by

$$\Big(\int_a^b F(\lambda)\, d\, E(\lambda)\Big)\varphi = \int_a^b F(\lambda)\big(d\, E(\lambda)\varphi\big).$$

Remark 1.10. *We notice that*

$$\begin{aligned}
\|\mathsf{S}(F,\pi,\{\xi_j\})\varphi\|^2 &= \Big(\mathsf{S}(\cdot,\cdot,\cdot)\varphi\Big|\mathsf{S}(\cdot,\cdot,\cdot)\varphi\Big) \\
&= \Big(\sum_{j=1}^n F(\xi_j)\, E(\Delta_j)\varphi\Big|\, \sum_{k=1}^n F(\xi_k)\, E(\Delta_k)\varphi\Big) \\
&= \sum_{j=1}^n \sum_{k=1}^n F(\xi_j)\overline{F(\xi_k)}\big(E(\Delta_j)\varphi\big|\, E(\Delta_k)\varphi\big) \\
&= \sum_{j=1}^n |F(\xi_j)|^2 \|E(\Delta_j)\varphi\|^2 \\
&= \sum_{j=1}^n |F(\xi_j)|^2 \big(E(\Delta_j)\varphi\big|\varphi\big).
\end{aligned}$$

Thus

$$\int_a^b F(\lambda) dE(\lambda)\varphi$$

exists if and only if

$$\underbrace{\int_a^b |F(\lambda)|^2\, d\|E(\lambda)\varphi\|^2}_{\|\int_a^b F(\lambda)dE(\lambda)\varphi\|^2} \qquad \textit{exists.}$$

Remark 1.11. *Let*

$$F(y) = \int_{-\infty}^{\infty} f(\lambda)\, d\big(E_\lambda y|x\big) \tag{1.13}$$

be a continuous linear form.

If

$$y = \int_a^b \overline{f(\lambda)}\, dE_\lambda x,$$

then

$$y = E_{(a,b]} y.$$

Indeed, it follows by using the Riemann sums and observing that

$$E_{(a,b]} E_{(\lambda_{j-1},\lambda_j]} = E_{(\lambda_{j-1},\lambda_j]}$$

and thus

$$E_{(a,b]}\Big(\sum_{j=1}^n \overline{f(\xi_j)}\,(E_{(\lambda_{j-1},\lambda_j]})x\Big) = \sum_{j=1}^n \overline{f(\xi_j)}\,(E_{(\lambda_{j-1},\lambda_j]})x.$$

Proposition 1.12. *Let f be a continuous function on $\mathbb{R}$ with complex values and let $x \in \mathcal{H}$. Then it is possible to define for $\alpha < \beta$, the integral*

$$\int_\alpha^\beta f(\lambda)\, dE_\lambda x$$

as the strong limit in $\mathcal{H}$ of the Riemann sum:

$$\sum_j f(\lambda'_j)\,(E_{\lambda_{j+1}} - E_{\lambda_j})x, \tag{1.14}$$

where $\alpha = \lambda_1 < \lambda_2 < \cdots < \lambda_n = \beta$, and $\lambda'_j \in (\lambda_j, \lambda_{j+1}]$, when $\max_j |\lambda_{j+1} - \lambda_j| \to 0$.

Proof. The proof uses the uniform continuity of f. $\square$

Definition 1.13. *For any given $x \in \mathcal{H}$ and any continuous function f on $\mathbb{R}$, the integral*

$$\int_{-\infty}^{\infty} f(\lambda)\, dE_\lambda x$$

is defined as the strong limit in $\mathcal{H}$, if it exists of $\int_\alpha^\beta f(\lambda)\, dE(\lambda)x$ when $\alpha \to -\infty$ and $\beta \to \infty$.

Remark 1.14. *The theory works more generally for any Borelian function see [3]. This is important, because we are in particular interested in the case when $f(\lambda) = \chi_{(-\infty,\lambda]}$.*

One possibility for the reader who wants to understand how this can be made is to look at [5] which gives the following theorem:

Theorem 1.15 ([5], Theorem 8.14, p. 173)**.**

(1) *If μ is a complex Borel measure on $\mathbb{R}$ and if*

$$f(x) = \mu((-\infty, x]), \quad \forall x \in \mathbb{R}, \tag{1.15}$$

then f is a normalized function with bounded variation(NBV), i.e. a function with bounded variation, which is continuous from the right and such that $\lim_{x \to -\infty} f(x) = 0$.

(2) *Conversely, to every $f \in NBV$, there corresponds a unique complex Borel measure μ such that* (1.15) *is satisfied.*

Theorem 1.16. *For x given in $\mathcal{H}$ and if f is a complex valued function on $\mathbb{R}$, the following conditions are equivalent*

(i)

$$\int_{-\infty}^{\infty} f(\lambda)\, dE_\lambda x \quad \textit{exists}; \tag{1.16}$$

(ii)

$$\int_{-\infty}^{\infty} |f(\lambda)|^2\, d\|E_\lambda\|^2 < \infty; \tag{1.17}$$

(iii)

$$y \mapsto \int_{-\infty}^{\infty} f(\lambda)\, d(\langle E_\lambda y, x\rangle_{\mathcal{H}}) \tag{1.18}$$

is a continuous linear form.

Sketch of the Proof.

(i) $\implies$ (iii) It follows by using repeatedly the Banach-Steinhaus Theorem and the definition of the integral.

(iii) $\implies$ (ii) Let F be a linear form defined in (1.18). Introducing

$$y = \int_{\alpha}^{\beta} \overline{f(\lambda)}\, dE_\lambda x,$$

we notice that

$$y = E_{(\alpha,\beta]} y$$

by using the Riemann integrals.

It is not difficult to show that

$$\begin{aligned}\overline{F(y)} &= \int_{-\infty}^{\infty} \overline{f(\lambda)}\, d\langle E_\lambda x, y\rangle \\ &= \int_{-\infty}^{\infty} \overline{f(\lambda)}\, d\langle E_\lambda x, E_{(\alpha,\beta]} y\rangle \\ &= \int_{-\infty}^{\infty} \overline{f(\lambda)}\, d\langle E_{(\alpha,\beta]} E_\lambda x, y\rangle \\ &= \int_{\alpha}^{\beta} \overline{f(\lambda)}\, d\langle E_\lambda x, y\rangle \\ &= \|y\|^2.\end{aligned}$$

By (1.16) it follows that

$$\|y\|^2 \le \|F\| \|y\|.$$

Thus

$$\|y\| \le \|F\|.$$

Observe that the right hand side is independent of α and β.

On the other hand, using once more the Riemann sums, we obtain

$$\|y\|^2 = \int_{\alpha}^{\beta} |\overline{f(\lambda)}|^2\, d\|E_\lambda x\|^2.$$

Therefore

$$\int_{\alpha}^{\beta} |\overline{f(\lambda)}|^2\, d\|E_\lambda x\|^2 \le \|F\|^2.$$

Thus, making $\alpha \to -\infty$ and $\beta \to \infty$ yield (1.17).

<u>(ii) $\implies$ (i)</u> Notice that for $\alpha' < \alpha < \beta < \beta'$, we

$$\begin{aligned}&\| \int_{\alpha'}^{\beta'} f(\lambda)\, dE_\lambda x - \int_{\alpha}^{\beta} f(\lambda)\, dE_\lambda x\|^2 \\ &= \int_{\alpha'}^{\alpha} |f(\lambda)|^2\, d\|E_\lambda x\|^2 + \int_{\beta}^{\beta'} |f(\lambda)|^2\, d\|E_\lambda x\|^2.\end{aligned}$$

This implies (1.15). □

Theorem 1.17. *Let $\lambda \mapsto f(\lambda)$ be a real-valued continuous function. Let*

$$D_f = \{x \in \mathcal{H} : \int_{-\infty}^{\infty} |f(\lambda)|^2 d\langle E(\lambda)x, x\rangle < \infty\}.$$

Then D_f is dense in $\mathcal{H}$ and we define T_f whose domain is defined by

$$D(T_f) = D_f,$$

and

$$\langle T_f x, y\rangle = \int_{-\infty}^{\infty} f(\lambda)\, d\langle E(\lambda)x, y\rangle$$

for all x in $D(T_f)$ and $y \in \mathcal{H}$.

The operator T_f is self-adjoint. In addition, $T_f E_\lambda$ is an extension of $E_\lambda T_f$.

Proof of Theorem. Property (1.2) gives us, that for any $y \in \mathcal{H}$, there exists a sequence (α_n, β_n) such that $E_{(\alpha_n,\beta_n]}y \to y$ as $n \to \infty$.

Observe that $E_{(\alpha,\beta]}y \in D_f$, for any α, β, this yields the density of D_f in $\mathcal{H}$.

Since f is real-valued and E_λ is symmetric, it follows that T_f is symmetric. That T_f is self-adjoint is deduced by using Theorem 1.16.

We notice that, for $f_0 = 1$, we get $T_{f_0} = I$ and for $f_1(\lambda) = \lambda$, we have a self-adjoint $T_{f_1} = T$.

In this case, it is said that

$$T = \int_{-\infty}^{\infty} \lambda\, dE(\lambda)$$

is a spectral decomposition of T and we observe that

$$\|Tx\|^2 = \int_{-\infty}^{\infty} \lambda^2\, d\langle E(\lambda)x, x\rangle = \int_{-\infty}^{\infty} \lambda^2\, d\|E(\lambda)x\|^2$$

for $x \in D(T)$. More generally,

$$\|Tx\|^2 = \int_{-\infty}^{\infty} \lambda^2\, d((E(\lambda)x, x)) = \int_{-\infty}^{\infty} \lambda^2\, d(\|E(\lambda)x\|^2)$$

for $x \in D(T_f)$. $\square$

We have seen so far how one can associate to a spectral family of projectors a self-adjoint operator.

The Spectral Decomposition Theorem makes explicit that the preceding situation is actually the general one.

Theorem 1.18. *Any self-adjoint operator T in a Hilbert space $\mathcal{H}$ admits a spectral decomposition such that*

$$\langle Tx, y\rangle = \int_{\mathbb{R}} \lambda \langle E_\lambda x, y\rangle_{\mathcal{H}}, \tag{1.19}$$

and

$$Tx = \int_{\mathbb{R}} \lambda\, d(E_\lambda x). \tag{1.20}$$

Sketch of the Proof.

Step 1. It is rather natural to imagine that it is essentially enough to treat the case when T is a bounded selfadjoint operator (or at least a normal bounded operator, that is satisfying $T^*T = TT^*$. If A is indeed a general semibounded self-adjoint operator, one can come back to the bounded case by considering $(A+\lambda_0)^{-1}$, with λ_0 real, which is bounded and self-adjoint. In the general case, one can consider $(A+i)^{-1}$.

Step 2. We analize first the spectrum of $P(T)$ where P is a polynomial.

Lemma 1.19. *If P is a polynomial, then*

$$\sigma(P(T)) = \{P(\lambda) : \lambda \in \sigma(T)\}.$$

Proof of Lemma 1.19. From the identity $P(x) - P(\lambda) = (x - \lambda)\, Q_\lambda(x)$ we obtain for bounded operators the identity

$$P(T) - P(\lambda) = (T - \lambda)\, Q_\lambda(T).$$

This allows us to construct the inverse of $(T - \lambda)$ if one knows the inverse of $P(T) - P(\lambda)$.

Reciprocally, notice that if $z \in \mathbb{C}$ and if $\lambda_j(z)$ are the roots of $\lambda \mapsto (P(\lambda) - z)$, then

$$(P(T) - z) = c \prod_j (T - \lambda_j(z)).$$

This allows to construct the inverse of $(P(T) - z)$ if one has the inverse of $(T - \lambda_j(z))$ for all j. $\square$

We will use the next exercises to prove the next lemma.

Exercise 1.20. *Let A be a bounded linear operator in a Hilbert space $\mathcal{H}$. Show that*

$$\|A^*A\| = \|A\|^2.$$

Exercise 1.21. *Let $A \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator. Show that the spectrum of A is contained in $[m, M]$ with $m = \inf \dfrac{\langle Au, u\rangle}{\|u\|^2}$ and $M = \sup \dfrac{\langle Au, u\rangle}{\|u\|^2}$. Moreover m and M belong to the spectrum of T.*

Lemma 1.22. *Let T be a bounded self-adjoint operator. Then we have*

$$\|P(T)\| = \sup_{\lambda \in \sigma(T)} |P(\lambda)|. \tag{1.21}$$

Proof. We first notice that

$$\|P(T)\|^2 = \|P(T)^* P(T)\|.$$

This is a consequence of the general property for bounded linear operators

$$\|A^* A\|^2 = \|A\|^2.$$

(See Exercise 1.20).

From Exercise 1.21 and Lemma 1.19 we deduce that

$$\begin{aligned}\|P(T)\|^2 &= \|(\overline{P}P)(T)\| \\ &= \sup_{\mu \in \sigma((\overline{P}P)(T))} |\mu| \\ &= \sup_{\lambda \in \sigma(T)} |(\overline{P}P)(\lambda)| \\ &= \sup_{\lambda \in \sigma(T)} |P(\lambda)|^2.\end{aligned}$$

□

Step 3. We have defined a map Φ from the set of polynomials into $\mathcal{B}(\mathcal{H})$ by

$$P \mapsto \Phi(P) = P(T) \tag{1.22}$$

which is continuous since

$$\|\Phi(P)\|_{\mathcal{B}(\mathcal{H})} = \sup_{\lambda \in \sigma(T)} |P(\lambda)|. \tag{1.23}$$

The set $\sigma(T)$ is a compact in $\mathbb{R}$ and using the Stone-Weierstrass theorem (which guarantees the density of the polynomials in $C(\sigma(T))$), the map Φ can be uniquely extended to $C(\sigma(T))$. We will denote this extension again using Φ.

Theorem 1.23 (Properties of Φ)**.** *Let T be a bounded self-adjoint operator on $\mathcal{H}$. Then there exists a unique map Φ,*

$$\Phi : C(\sigma(T)) \to \mathcal{B}(\mathcal{H})$$

satisfying the following properties:

(i)

$$\begin{aligned}&\Phi(f+g) = \Phi(f) + \Phi(g); \\ &\Phi(\lambda f) = \lambda \Phi(f); \\ &\Phi(1) = \mathrm{I}_d; \\ &\Phi(\bar{f}) = \Phi(f)^*; \\ &\Phi(fg) = \Phi(f) \circ \Phi(g).\end{aligned}$$

(ii)

$$\|\Phi(f)\|_{\mathcal{B}(\mathcal{H})} = \sup_{\lambda \in \sigma(T)} |f(\lambda)|.$$

(iii) *If f is defined by $f(\lambda) = \lambda$, then $\Phi(f) = T$.*

(iv)

$$\sigma(\Phi(f)) = \{f(\lambda) : \lambda \in \sigma(T)\}.$$

(v) *If φ satisfies $T\varphi = \lambda\,\varphi$, then $\Phi(f)\varphi = f(\lambda)\,\varphi$.*

(vi) *If $f \geq 0$, then $\Phi(f) \geq 0$.*

Proof. The proof of the properties above follows by showing first the properties for the polynomials P and then extending the properties by continuity to continuous functions. To establish the last item we observe that

$$\Phi(f) = \Phi(\sqrt{f}) \cdot \Phi(\sqrt{f}) = \Phi(\sqrt{f})^* \cdot \Phi(\sqrt{f}).$$

$\square$

Step 4. Now we introduce the measures.

Let $\psi \in \mathcal{H}$. Define the functional

$$f \mapsto \langle \psi, f(T)\psi \rangle_{\mathcal{H}} = \langle \psi, \Phi(f)\psi \rangle_{\mathcal{H}}. \tag{1.24}$$

We observe that this is a positive linear functional on $C(\sigma(T))$. From the Riesz Theorem (Theorem 1.32 below), there exists a unique measure μ_ψ on $\sigma(T)$, such that

$$\langle \psi, \Phi(f)\psi \rangle_{\mathcal{H}} = \int_{\sigma(T)} f(\lambda)\, d\mu_\psi. \tag{1.25}$$

This measure is called **the spectral measure** associated with the vector $\psi \in \mathcal{H}$. This measure is a Borel measure. This means that we can extend the map Φ and (1.25) to Borelian functions.

Using the standard Hilbert calculus (that is the link between sesquilinear form and the quadratic forms) we can also construct for any x and y in $\mathcal{H}$ a complex measure $d\mu_{x,y}$ such thta

$$\langle x, \Phi(f)y \rangle_{\mathcal{H}} = \int_{\sigma(T)} f(\lambda)\, d\mu_{x,y}(\lambda). \tag{1.26}$$

Using the Riesz representation Theorem (Theorem 1.33 below), this gives us, when f is bounded, an operator $f(T)$. If $f = \chi_{(-\infty,\mu]}$, we recover the operator $E_\mu = f(T)$ which permits to construct indeed the spectral family announced in Theorem 1.18.

$\square$

Remarks 1.24. *For any measurable (real or complex valued) function f on $\mathbb{R}$, the unique operator $f(T)$ satisfying* (1.25) *is defined. Its domain is the set $\{h : \int_{-\infty}^{\infty} |f|^2 \, d\mu_h < \infty\}$, dense in $\mathcal{H}$.*

For any $h \in D(f(T))$

$$\|f(T)h\|^2 = \int_{-\infty}^{\infty} |f(\lambda)|^2 \, d\mu_h(\lambda). \tag{1.27}$$

The equation (1.27) *can be easily verified for the case when f is a nonnegative measurable function.*

We have

$$\begin{aligned} \|f(T)h\|^2 &= \lim_{n\to\infty} \|f \wedge n \cdot \chi_{[-n,n]}(T)h\|^2 \\ &= \lim_{n\to\infty} \big(\big[f \wedge n \cdot \chi_{[-n,n]}(T) \big]^2 h, h \big) \\ &= \lim_{n\to\infty} \int_{-\infty}^{\infty} \big[f \wedge n \cdot \chi_{[-n,n]}(\lambda) \big]^2 \, d\mu_h(\lambda) \\ &= \int_{-\infty}^{\infty} f^2 \, d\mu_h(\lambda) \end{aligned}$$

where $f \wedge n \cdot \chi_{[-n,n]} = \inf\{f, n \cdot \chi_{[-n,n]}\}$.

In case f is any measurable function, $f = f_1 - f_2 + i(g_3 - g_4)$ and $|f|^2 = f_1^2 + f_2^2 + g_3^2 + g_4^2$. In this situation equation (1.27) *can be seen to hold.*

1.3. **Another version of the Spectral Theorem.** In this section our goal is to present a multiplication form of the Spectral Theorem. Our plan is to sketch the main points of the proof of the theorem. We will ask the reader to complete some details by proving some proposed exercises.

The Spectral Theorem reads as follows.

Theorem 1.25 (Multiplication Operator Form of the Spectral Theorem)**.** *Let T be a self-adjoint operator in a Hilbert space $\mathcal{H}$. Then there exist a measure space $(X, \mathcal{A}, \mu)$, a unitary operator $U : \mathcal{H} \to L^2(X, \mu)$, and a measurable function F on X which is real a.e. such that*

(i) *$h \in D(T)$ if and only if $F(\cdot)Uh(\cdot)$ is in $L^2(X, \mu)$ and*

(ii) *if $f \in U(D(T))$, then $(UTU^{-1}f)(\cdot) = F(\cdot)f(\cdot)$.*

To prove this theorem we need of some preparation.

Definition 1.26. *Let $T : \mathcal{H} \to \mathcal{H}$ be a continuous linear operator with adjoint T^*.*

T is called **normal** *if and only if $T^*T = TT^*$.*

The proof of Theorem 1.25 uses the following Spectral Theorem for bounded normal linear operators. The proof can be found for instance in the appendix of [2].

Theorem 1.27. *Let $T = T_1 + iT_2$ be a bounded normal operator on $\mathcal{H}$. Then there exist a family of finite measures $(\mu_j)_{j\in I}$ on $\sigma(T_1) \times \sigma(T_2)$ and a unitary operator*

$$U : \mathcal{H} \to \bigoplus_{j\in I} L^2(\sigma(T_1) \times \sigma(T_2), \mu_j)$$

such that

$$(UTU^{-1}f)_j(x, y) = (x + iy)f_j(x, y) \quad a.e.$$

where $f = (f_j)_{j\in I}$ is in $\bigoplus_{j\in I} L^2(\sigma(T), \mu_j)$ and $\sigma(\cdot)$ stands for the spectrum of the operator $\cdot$.

Proof. See Theorem A.6 in [2]. $\square$

A readily consequence of this theorem we have.

Corollary 1.28. *Let T be a bounded normal operator on a Hilbert space $\mathcal{H}$.Then there exists a measure space $(X, \mathcal{A}, \mu)$, a bounded complex function G on X, and a unitary map $U : \mathcal{H} \to L^2(X, \mu)$ so that*

$$(UTU^{-1}f)(\lambda) = G(\lambda)f(\lambda) \quad a.e.$$

We will use the following facts in the proof of Theorem 1.25.

Exercise 1.29. *Show that if T is a closed linear operator in $\mathcal{H}$ densely defined and $\lambda \in \rho(T)$, then $(T - \lambda I)^{-1}$ is a bounded linear operator on $\mathcal{H}$.*

Exercise 1.30. *Let T be a self-adjoint operator in $\mathcal{H}$. Prove that $\rho(T)$ contains all complex number with nonzero imaginary part. Moreover, if $\operatorname{Im}\lambda \neq 0$, then*

$$\|(T - \lambda I)^{-1}\| \leq \frac{1}{|\operatorname{Im}\lambda|} \tag{1.28}$$

and

$$\operatorname{Im}\langle (T - \lambda I)h, h\rangle = \operatorname{Im}(-\lambda)\|h\|^2 \quad \text{for all} \quad h \in D(T). \tag{1.29}$$

Proof of Theorem 1.25. We will need the results in Corollary 1.28 for bounded normal operators applying to the operator $(T+i)^{-1}$.

We first show that $(T+i)^{-1}$ is a bounded normal operator. From the Exercises 1.29 and 1.30 we conclude that $(T \pm i)^{-1}$ exists as bounded linear operator in $\mathcal{H}$. In particular, $R(T \pm i) = \mathcal{H}$ and $T \pm i$ are one-to-one operators. Since T is self-adjoint, for any ϕ and ψ in $D(T)$, we have

$$((T-i)\phi, (T+i)^{-1}(T+i)\psi) = ((T-i)^{-1}(T-i)\phi, (T+i)\psi).$$

This implies that $((T+i)^{-1})^* = (T-i)^{-1}$. Since $(T+i)^{-1}$ and $(T-i)^{-1}$ commute by the resolvent formula, we have

$$(T+i)^{-1}((T+i)^{-1})^* = (T+i)^{-1}(T-i)^{-1} = ((T+i)^{-1})^*(T+i)^{-1},$$

which tells us that $(T+i)^{-1}$ is a normal operator.

Using Corollary 1.28, there is measure space $(X, \mathcal{A}, \mu)$, a unitary operator $U : \mathcal{H} \to L^2(X, \mu)$, and a bounded, measurable complex function G on X such that

$$(U(T+i)^{-1}U^{-1}f)(x) = G(x)f(x) \quad \text{a.e.} \tag{1.30}$$

for all $f \in L^2(X, \mu)$.

Since $Ker(T+i)^{-1} = \{0\}$, $G(x) \neq 0$ a.e. Therefore if we define $F(x)$ as $G(x)^{-1} - i$ for each $x \in X$, $|F(x)|$ is finite a.e. Now if $f \in U(D(T))$, then there exists a function $g \in L^2(X, \mu)$ such that $f(\cdot) = G(\cdot)g(\cdot)$ in L^2. This is true because of

$$U(D(T)) \subset U(T+i)^{-1}(\mathcal{H}) \subset U(T+i)^{-1}U^{-1}(L^2(X, \mu)). \tag{1.31}$$

Noticing that $U(T+i)^{-1}U^{-1}$ is an injection, for any g in the range of $U(T+i)^{-1}U^{-1}$ we have from (1.30) that

$$[U(T+i)^{-1}U^{-1}]^{-1}g(x) = \frac{1}{G(x)} \cdot g(x) \in L^2(X, \mu).$$

In particular for f in the set $U(D(T))$,

$$[U(T+i)^{-1}U^{-1}]^{-1}f(x) = \frac{1}{G(x)} \cdot f(x) \in L^2(X, \mu).$$

or

$$U(T+i)U^{-1}f(x) = \frac{1}{G(x)} \cdot f(x) \in L^2(X, \mu).$$

or

$$UTU^{-1}f(x) = \frac{1}{G(x)} \cdot f(x) - if(x) = F(x)f(x) \in L^2(X, \mu).$$

This proves (ii) and the necessity of (i) provided F is real-valued, which we show below. For the converse of (i), if $F(x)Uh(x)$ is in

$L^2(X,\mu)$, then there exists $k \in \mathcal{H}$ so that $Uk = [F(x)+i]Uh(x)$. Thus

$$G(x)Uk(x) = G(x)\,[F(x)+i]Uh(x) = Uh(x),$$

so $h = (T+i)^{-1}k$, whereby $h \in D(T)$.

To finish the proof it must be established that F is real-valued a.e. Observe that the operator in $L^2(X,\mu)$ defined by multiplication by F is self-adjoint since by (ii) it is unitarily equivalent to T. Hence for all χ_M, M a measurable subset of X, $(\chi_M, F\chi_M)$ is real. However, if $\operatorname{Im} F > 0$ on a set of positive measure, then there exists a bounded set B in the plane so that $M = f^{-1}(B)$ has nonzero measure. Clearly $F\chi_M$ is in $L^2(X,\mu)$ since B is bounded and $\operatorname{Im}(\chi_M, F\chi_M) > 0$. This contradiction shows that $\operatorname{Im} F = 0$ a.e. □

Example 1.31 (Examples of functions of a self-adjoint operator).
The following are common examples in spectral theory.

(1) *f is the characteristic function of $(-\infty,\lambda]$, $\chi_{(-\infty,\lambda]}$; $\Phi(f) = f(T)$ is then $\Phi(f) = E(\lambda)$.*
(2) *f is the characteristic function of $(-\infty,\lambda)$, $\chi_{(-\infty,\lambda)}$; $f(T)$ is then $\Phi(f) = E(\lambda - 0)$.*
(3) *f is a compactly supported continuous function. $f(T)$ will be an operator whose spectrum is localized in the support of f.*
(4) *$f_t(\lambda) = \exp(it\lambda)$ with t real. $f_t(T)$ is then a solution of the functional equation*

$$\begin{cases} (\partial_t - iT)(f(t,T)) = 0, \\ f(0,T) = I. \end{cases}$$

We notice that , for all real t, $f_t(T) = \exp(itT)$ is a bounded unitary operator.
(5) *$g_t(\lambda) = \exp(-t\lambda)$ with t real positive. $g_t(T)$ is the a solution of the functional equation*

$$\begin{cases} (\partial_t + T)(g(t,T)) = 0, \quad \textit{for } t \geq 0, \\ g(0,T) = I. \end{cases}$$

1.4. **Application of the Spectral Theorem in solving the Schrödinger equation.** The time-dependent Schrödinger equation arises in quantum mechanics. It is given by

$$i\frac{du}{dt} = A\,u(t),$$

where $u(t)$ is an element of a Hilbert space $\mathcal{H}$, A is a self-adjoint operator in $\mathcal{H}$, and t is a time variable with $u(t) \in D(A)$. An initial condition is $u(0) = u_0 \in D(A)$. The derivative of u is given as

$$\lim_{\Delta \to 0} \frac{u(t+\Delta) - u(t)}{\Delta}$$

in the strong topology of $\mathcal{H}$.

The Spectral Theorem allows us to solve the Schrödinger equation. Let e^{-itA} be the bounded operator on $\mathcal{H}$ given by

$$e^{-itA} = \int_{-\infty}^{\infty} e^{-it\lambda}\, dP(\lambda),$$

where $A = \displaystyle\int \lambda\, dP$. We would like to prove that

$$\frac{d}{dt}(e^{-itA}h) = i\,A(e^{-itA}h) \tag{1.32}$$

for every $h \in D(A)$.

To show this, we compute the following limit:

$$\begin{aligned}
&\lim_{\Delta t \to 0}\Big\|\Big(\frac{e^{-i(t+\Delta t)A} - e^{-itA}}{\Delta t} + ie^{-itA}A\Big)h\Big\|^2 \\
&\qquad = \lim_{\Delta t \to 0}\int_{-\infty}^{\infty}\Big|\frac{e^{-i(t+\Delta t)\lambda} - e^{-it\lambda}}{\Delta t} + ie^{-it\lambda}\lambda\Big|^2 d(E(\lambda)h,h) \\
&\qquad = \lim_{\Delta t \to 0}\int_{-\infty}^{\infty}\Big|\frac{e^{-i\Delta t\lambda} - 1}{\Delta t} + i\lambda\Big|^2 d(E(\lambda)h,h).
\end{aligned}$$

Using that $|e^{ix} - 1| \leq |x|$, the integrand above is bounded by $4\lambda^2$, which is integrable since $h \in D(A)$. It follows then by using the Lebesgue Dominated Convergence theorem that the limit is zero. Hence

$$\frac{d}{dt}(e^{-itA}h) = -i\,(e^{-itA}Ah) \tag{1.33}$$

for every $h \in D(A)$.

The identity (1.32) is obtained from (1.33) since for $h \in D(A)$

$$e^{-itA}Ah = Ae^{-itA}h. \tag{1.34}$$

This last identity follows from the fact that if $h \in D(A)$, then $e^{-itA}h$ is in $D(A)$ since by the equation (1.27) we have

$$\|E(M)e^{-itA}h\|^2 = \int \chi_M |e^{-it\lambda}|^2 \, dE_h(\lambda) = \int \chi_M dE_h(\lambda) = \|E(M)h\|^2.$$

The solution $u(t) = e^{-itA}u_0$ of the Schrödinger equation is unique. To show this, suppose that $v(t)$ in $D(A)$ is a solution. Then for any $\phi \in \mathcal{H}$

$$\begin{aligned}
\frac{d}{ds}(e^{-i(t-s)A}v(s), \phi) &= \lim_{\Delta s \to 0} \frac{(e^{-i(t-(s+\Delta s))A}\, v(s+\Delta s), \phi) - (e^{-i(t-s)A}v(s), \phi)}{\Delta s} \\
&= \lim_{\Delta s \to 0} \Big(\frac{e^{-i(t-(s+\Delta t))A} - e^{-i(t-s)A}}{\Delta s} v(s+\Delta s), \phi \Big) \\
&\quad + \Big(\lim_{\Delta s \to 0} e^{-i(t-s)A} \frac{v(s+\Delta s) - v(s)}{\Delta s}, \phi \Big) \\
&= \Big(-\frac{d}{dt}\, e^{-i(t-s)A} v(s), \phi \Big) + \Big(e^{-i(t-s)A}\, \frac{dv}{ds}, \phi \Big) \\
&= (ie^{-i(t-s)A}\, v(s), \phi) + (e^{-i(t-s)A}[-iAv(s)], \phi) = 0.
\end{aligned}$$

Therefore for all $\phi \in \mathcal{H}$

$$0 = \int_0^t \frac{d}{ds}\big(e^{-i(t-s)A}\, v(s), \phi\big)\, ds = (e^{-i0A}v(t), \phi) - (e^{-itA}v(0), \phi),$$

and since $v(0) = u_0$ and $e^{-i0A} = I$ we have

$$v(t) = e^{-itA}u_0.$$

This yields the uniqueness.

1.5. **Riesz representation Theorem.** We start by introducing some notations and definitions.

Given a locally compact Hausdorff space X we denote $C_0(X)$ as the set of continuous functions on X which vanish at infinity.

We say that ν is a **regular** measure if every Borel set in X is both outer regular and inner regular. We denote by $|\nu|$ the **total variation** of ν or the **total variation measure**.

A complex Borel measure μ on X is called regular if $|\mu|$ is regular.

If μ is a complex Borel measure on X, it is not difficult to see that the mapping

$$f \to \int_X f \, d\mu$$

is a bounded linear functional on $C_0(X)$, whose norm is not longer than $|\mu|(X)$. The Riesz theorem guarantees that all bounded linear functionals on $C_0(X)$ are obtained in this way.

Theorem 1.32. *If X is a locally compact Hausdorff space, then every bounded linear functional Φ on $C_0(X)$ is represented by a unique regular complex Borel measure μ, in the sense that*

$$\Phi f = \int_X f \, d\mu \quad \text{for every } f \in C_0(X).$$

Moreover, the norm of Φ is the total variation of μ:

$$\|\Phi\| = |\mu|(X).$$

Proof. See Theorem 6.19 in [5]. □

In Hilbert spaces we have the well known Riesz theorem.

Theorem 1.33. *Let $u \mapsto F(u)$ a linear continuous form on $\mathcal{H}$. Then there exists a unique $w \in \mathcal{H}$ such that*

$$F(u) = \langle u, w \rangle_{\mathcal{H}}, \quad \forall u \in \mathcal{H}. \tag{1.35}$$

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